Traveling Waves of Fisher-KPP Equation by Direct Similarity Method

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Abstract: Fisher-KPP equation is subjected to direct similarity method to determine its traveling waves as invariant solutions. All the computations are carried out using the software WOLFRAM MATHEMATICA Version Number:12.2.0.0.

1. Introduction

Let $u(x,t)$ be the frequency of the mutant genes in a linear habitat which is distributed with uniform density. When the inevitable mutation takes place in a gene, known hereafter as the advantageous gene, the population of these mutant genes increase at the expense of the allelomorphs. If m is the intensity of the selection in favor of the mutant genes and k the coefficient of diffusion of the mutant genes away from the location of their parent allelomorphs. Fisher (1937) and Kolmogorov, Petrovsky and Piscounov (1937) modeled this diffusion process into a (1+1)-dimensional nonlinear partial differential equation, known as, Fisher-KPP equation

$$u_t = k\ u_{xx} + m\ u\ (1-u), \tag{1}$$

By a suitable scaling the constants k, m be set equal to unity. Equation (1) arises not only in ecology but also in biology and heat and mass transfer.

Ablowitz and Zeppetella (1979) looked for traveling wave solutions of (1), with $k = m = 1$, in the form $u(x,t) = u(z), z = x - ct$. Apparently $u(z)$ is governed by the nonlinear differential equation

$$\frac{d^2u}{dz^2} + c\frac{du}{dz} + u - u^2 = 0, \tag{2}$$

with the solution

$$u(z) = \left(1 - Ae^{\frac{z}{\sqrt{6}}}\right)^{-2}, z = x - \frac{5}{\sqrt{6}}t, \tag{3}$$

Here A is a free constant.

For

$$u_t = u_{xx} + u - u^k, \tag{4}$$

Kaliappan (1984) found

$$\left(1 - Ae^{\frac{k-1}{\sqrt{2(k+1)}}z}\right)^{-2/(k-1)}, \tag{5}$$

as its solution by applying the regular perturbation method (Zwillinger (1997)). Note that when $k = 2$, the solution (5) reduces to (3) as expected.

In this paper we apply the direct similarity method to (2) to derive its traveling wave solutions. It is customary that the derivation of invariant solutions is carried out using the Lie's Classical Method (Ovsiannikov (1982), Bluman and Kumei (1989) and Olver (1995)). In order to explore the invariance analysis the non-classical method of Bluman and Cole (1969) and the direct similarity method of Clarkson and Kruskal (1989) are recommended.

It is to be remarked that we do not make use of the third Remark, which facilitates the determination of the similarity variable, of the direct method. Further the computations are made simple by the use of WOLFRAM MATHEMATICA Version Number:12.2.0.0.

II. Solutions by Direct Similarity Method

Consider the Fisher-KPP equation

$$w_t = w_{xx} + a\, w\,(1 - w), \tag{6}$$

where a is a constant. We seek solutions of (6) in the form of

$$w(x,t) = A(x,t) + B(x,t)F(\xi), \quad \xi = \xi(x,t), \tag{7}$$

Substitute (7) in (6) we get an ordinary differential equation for $F(\xi)$:

$$\Gamma_1 + \Gamma_2 F + \Gamma_3 F' + \Gamma_4 F^2 + F'' = 0. \tag{8}$$

where the four functions $\Gamma_n(\xi),\ n = 1,2,3,4$ are introduced according to

$$aA - aA^2 - A_t + A_{xx} = B\xi_x^2\Gamma_1, \tag{9}$$
$$aB - 2aAB - B_t + B_{xx} = B\xi_x^2\Gamma_2, \tag{10}$$
$$B\xi_t + 2B_x\xi_x + B\xi_{xx} = B\xi_x^2\Gamma_3, \tag{11}$$
$$-aB^2 = B\xi_x^2\Gamma_4. \tag{12}$$

We shall now solve (9) – (12) for $A(x,t), B(x,t), \xi(x,t)$ and $\Gamma_n(\xi), n = 1,2,3,4$ with the help of the following two remarks:

Remark 1: If $A(x,t)$ has the form $A(x,t) = \hat{A}(x,t) + B(x,t)\Omega(\xi)$ then we may choose $\Omega \equiv 0$

Remark 2: If $B(x,t)$ is found to have the form $B(x,t) = \hat{B}(x,t)\Omega(\xi)$ then we may put $\Omega \equiv 1$.

With $\Gamma_4 = -a\Omega_4$ equation(12) reads as $B = \xi_x^2\Omega_4$; in view of Remark 2 we take $\Omega_4 = 1$ ($\Gamma_4 = -a$) and finally find B as

$$B = \xi_x^2. \tag{13}$$

We insert (13) into (11) and assume $\Gamma_3 = 0$ to obtain the following second order linear heat conduction equation governing the similarity variable ξ:

$$\xi_t - 5\xi_{xx} = 0. \tag{14}$$

The linear partial differential equation (14) can be replaced with

$$\eta'' - \alpha\eta' = 0, \tag{15}$$

through the transformation

$$\xi(x,t) = \eta(z), \quad z(x,t) = x - 5\alpha t, \tag{16}$$

where α is a real positive constant. A solution of (15) is

$$\eta(z) = e^{-\alpha z}. \tag{17}$$

Now equations (16) and (17) yield

$$\xi(x,t) = e^{-\alpha(x-5\alpha t)} \quad \text{or} \quad \xi = e^{5\alpha^2 t - \alpha x}, \tag{18}$$

Equation (10) yields

$$A - \frac{1}{2} + \frac{B_t}{2aB} - \frac{B_{xx}}{2aB} = B\frac{\Gamma_2}{-2a}. \tag{19}$$

Remark 1 requires that $\Gamma_2 = 0$ and we have

$$A = \frac{1}{2} - \frac{3\alpha^2}{a}, \tag{20}$$

where we substituted for B, ξ and their derivatives using (13) and (18). Equations (9) and (20), with $\Gamma_1 = 0$, lead to

$$\alpha^2 = \pm\frac{a}{6} \tag{21}$$

Case-1: $\alpha = \sqrt{-\frac{a}{6}}$

In view of (21), equations (13), (18) and (19) become

$$A = 1, \; B = -\frac{a}{6}e^{-\frac{5a}{3}\; t - \sqrt{-\frac{2a}{3}}\; x}, \; \xi = e^{-\frac{5a}{6}\; t - \sqrt{-\frac{a}{6}}\; x}. \tag{22}$$

Substituting for $\Gamma_n(\xi), \; n = 1,2,3,4$, equation (8) simplifies to

$$F''(\xi) - aF(\xi)^2 = 0. \tag{23}$$

With $a = 6b^3$, the general solution of the nonlinear equation is

$$F(\xi) = \frac{1}{b}WeierstrassP\left[(b(\xi + c_1), \{0, c_2\})\right]. \tag{24}$$

Inserting from (22) and (24) into (7) we have a solution of (7) in terms of Weierstrass function

$$w(x,t) = 1 - \frac{a}{6}e^{-\frac{5a}{3}\; t - \sqrt{-\frac{2a}{3}}\; x}\frac{1}{b}WeierstrassP\left[\left(b(e^{-\frac{5a}{6}\; t - \sqrt{-\frac{a}{6}}\; x} + c_1), \{0, c_2\}\right)\right], \tag{25}$$

where a and b are related through $a = 6b^3$.

Case-2: $\alpha = \sqrt{\frac{a}{6}}$

We here have

$$\alpha^2 = \frac{a}{6} \quad \text{or} \quad \alpha = \sqrt{\frac{a}{6}}. \tag{26}$$

In view of (26), equations (13), (18) and (19) become

$$A = 0, \; B = \frac{a}{6}e^{\frac{5a}{3}\; t - \sqrt{\frac{2a}{3}}\; x}, \; \xi = e^{\frac{5a}{6}\; t - \sqrt{\frac{a}{6}}\; x}. \tag{27}$$

Substituting for $\Gamma_n(\xi),\ n = 1, 2, 3, 4$, equation (8) simplifies to

$$F''(\xi) - aF(\xi)^2 = 0. \tag{28}$$

With $a = 6b^3$, the general solution of the nonlinear equation (28) is

$$F(\xi) = \frac{1}{b} WeierstrassP\left[(b(\xi + c_3), \{0, c_4\})\right]. \tag{29}$$

Inserting from (27) and (29) into (7) we have a solution of (6) in terms of Weierstrass function

$$w(x,t) = \frac{a}{6} e^{\frac{5a}{3}\ t - \sqrt{\frac{2a}{3}}\ x} \frac{1}{b} WeierstrassP\left[\left(b(e^{\frac{5a}{6}\ t - \sqrt{\frac{a}{6}}\ x} + c_3), \{0, c_4\}\right)\right], \tag{30}$$

We remark that (Polyanin and Zaitsev (2004))

$$WeierstrassP\left[\eta, \{0, 0\}\right] = \eta^{-2}. \tag{31}$$

Now if we set in (25), $c_1 = 1/C$ and $c_2 = 0$ then (30) reduces to

$$w(x,t) = \frac{1 + 2Ce^{-\frac{5a}{6}\ t - \sqrt{-\frac{a}{6}}\ x}}{(1 + Ce^{-\frac{5a}{6}\ t - \sqrt{-\frac{a}{6}}\ x})^2}. \tag{32}$$

Again if we set in (30), $c_3 = C$ and $c_4 = 0$ then (32) reduces to

$$w(x,t) = \frac{1}{(1 + Ce^{-\frac{5a}{6}\ t + \sqrt{\frac{a}{6}}\ x})^2}, \tag{33}$$

where C is an arbitrary constant. (Cf. Danilov, Maslov, and Volosov (1995), Kudryashov (1993))

III. Results and Conclusions

Direct similarity method is applied to the Fisher-KPP equation is

$$w_t \ = \ w_{xx} + a\ w\ (1 - w), \tag{34}$$

where a is a constant, to reduce it to an equation (28). As there are two cases the solutions in terms of Weierstrass P functions are presented for each case. Known special solutions, namely, (32) and (33) have been deduced for a suitable choices of the parameters appearing in these two solutions. We conclude that the direct similarity method is efficient in the derivation of the solutions of the nonlinear Fisher-KPP equation, and, in general, nonlinear equations. Of course one needs to tract the computation using MATHEMATICA.

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