Super Triangular Graceful Labeling of Bistar Graphs

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Abstract : Graph labeling is an active area in graph theory. Let $G = (V, E)$ be given graph. A graph labeling of G is an assignment of integers or group of elements to the elements of V, E, or $V \cup E$ satisfying certain prescribed properties. The vertex-labeling together with its induced edge-labeling we will call simply a labeling of G. Vast amount of literature is available on different types of graph labeling. In this paper, we extend studies on the properties and characteristics of super triangular graceful labeling.

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1. Introduction

Graph theory is a branch of Mathematics which deals the problems, with the help of diagrams. There are may applications of graph theory to a wide variety of subjects which include Operations Research, Physics, Chemistry, Computer Science and other branches of science.

The study of graph labelings has been and continues to be a popular topic of graph theory. If the vertices or edges or both of the graph are assigned values subject to certain condition(s) then it is known as *graph labeling*.

A simple graph $G(n, q)$ with n vertices and q edges is *graceful* if there is a labeling f of its vertices with distinct integers from the set $\{0, 1, 2, ..., q\}$ so that the induced edge labeling f', defined by $f'(uv) = |f(u) - f(v)|$ assigns each edge a different label.

Let T_q be the $q-th$ triangular number of the triangular series $T_1 = 1, T_2 = 3, T_3 = 6, ..., T_n = \frac{n(n+1)}{2}, ...$ A simple graph $G(n, q)$ with n vertices and q

edges is *triangular graceful* if there is an injection f from $V(G)$ to $\{0, 1, 2, ..., T_q\}$ such that the labels induced on each edge uv by $|f(u) - f(v)|$ are the first q triangular numbers.

Let S_q be the $q-th$ super triangular number of the super triangular series $S_1 = 1, S_2 = 5, S_3 = 14, ..., S_n = \frac{n(n+1)(2n+1)}{6}, ...$ A simple graph $G(n, q)$ with n vertices and q edges is super triangular graceful labeling if there is an injective function $f : V(G) \to \{0,1,2,...,S_q\}$ such that the induced edge labeling $f^*(u,v) = |f(u) - f(v)|$ is a bijection onto the set $\{S_1, S_2, ..., S_q\}$.

In this paper, we extend the concept of super triangular graceful labeling. It is shown that $B_{n,n}$, $\langle B_{n,n} : w \rangle$ and $\langle B_{n,n} : P_m \rangle$ are super triangular graceful graph. Here, the finite, simple and undirected graphs are considered.

2. Literature review

Rosa [1] introduced a new graph labeling called β - *labeling* in which the vertices are labeled with distinct numbers chosen from 0 to q, where q is the number of edge, such that each edge is labeled with the absolute difference of the labels of its end vertices and it is unique in the graph. He defined an α *-labeling* as a graceful labeling with the additional property that there exists an integer k so that for each edge xy either $f(x) \le k < f(y)$ or $f(y) \le k < f(x)$. He also defined another type of labeling such as σ -labeling. This is permitting the vertices of a graph with q edges to assume labels from the set $\{0, 1, 2, ..., 2q\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, ..., q\}$.

Rosa [1] invented another analogue of graceful labeling such as $\hat{\rho}$ - *labeling*. This is permitting the vertices of a graph with q edges to assume labels from the set $\{0, 1, 2, ..., q+1\}$, while the edge labels induced by the absolute value of the difference of the vertex labels are $\{1, 2, ..., q-1, q\}$ or $\{1, 2, ..., q-1, q+1\}$. Frucht [2] used the term nearly graceful labeling instead of $\hat{\rho}$ -labelings. A $\hat{\rho}$ -labeling is called *nearly graceful labeling* if

$E(G) = \{1, 2, ..., q-1, q+1\}$.

A few years later, Golomb [3] renamed β - labeling as *graceful labeling* as it is known today. Alternatively, Buratti et al. [4] defined a graph G with q edges to be *graceful* if there is an injection f from the vertices of G to the set $\{0, 1, 2, ..., q\}$ such that every possible difference of the vertex labels of all the edges is the set $\{1, 2, ..., q\}$.

Chartrand et al. [5] defined the concept of γ -labeling. Suppose that G is a graph with q edges. Then a γ - *labeling* is a injection f from $V(G)$ to the integers $0, 1, 2, ..., q$ such that induces an edge labeling f' defined by $f'(uv) = |f(u) - f(v)|$ for each edge uv. They defined the following parameters of a γ -labeling: $val(f) = \sum f'(e)$ over all edges e of G;

$$val_{\max}(G) = \max\{val(f) : f \text{ is a } \gamma - \text{labeling of } G\},$$

$$val_{\min}(G) = \min\{val(f) : f \text{ is a } \gamma - \text{labeling of } G\}.$$

A ρ *-labeling* of a graph is an injection from the vertices of a graph with q edges to the set $\{0, 1, 2, ..., 2q\}$ with the property that if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, ..., a_q$, then $a_i = i$ or $a_i = 2q+1-i$. Jesintha and Hilda [6] introduced a variation of Rosas ρ -labeling such as $\rho*$-labeling. A $\rho*$-*labeling* of a graph is an injection from the vertices of a graph with q edges to the set $\{0, 1, 2, ..., 2q\}$ with the property that if the edge labels induced by the absolute value of the difference of the vertex labels are $a_1, a_2, ..., a_q$, then $a_i = i$ or $a_i = 2qi$.

Moulton [7] proved Rosa's conjecture while introducing the concept of *almost graceful*. This is permitting the vertex labels to come from $\{0, 1, 2, ..., q-1, q+1\}$ while the edge labels are $\{1, 2, ..., q-1, q\}$ or $\{1, 2, ..., q-1, q+1\}$. Frucht [2] introduced the concept of pseudo graceful labeling. If G is a finite simple undirected graph with p vertices and $q = p-1$ edges, then they call a almost graceful labeling is *pseudo graceful* if $E(G) = \{1, 2, ..., q\}$.

Sekar [8] defined a concept of *one modulo three graceful labeling*. This is an injection f from the vertices of a graph with q edges to the set $\{0, 1, 3, 4, 6, 7, ..., 3(q-1), 3q-2\}$ and the labels induced on each edge uv by $|f(u) - f(v)|$ are $1, 4, 7, ..., 3q-2$.

Let $G=(V,E)$ be a graph with q edges. Let $f:V(G)\rightarrow\{0,1,2,...,q+k-1\}$ be a injection. Then the labeling f is said to be k*-graceful* if the edges of G can be labeled by the absolute value of the difference of the labels of adjacent vertices is $\{k,k+1,...,q+k-1\}$. If the graph G admits such a labeling, then G is said to be a k-graceful graph. These graphs introduced independently by Slater [9] and by Maheo and Thuillier [10]. Obviously, 1-graceful is graceful. Graphs that are k-graceful for all k are sometimes called *arbitrarily graceful*.

Kaneria and Makadia [11] investigated a new graph which is called swastik graph. They proved that the swastik graph is graceful. Vaidya and Shah [12] proved that the square graph of bistar, splitting graph of bistar and the splitting graph of star are graceful graphs. Kaneria et al. [13] proved that the path union of complete bipartite graph and join sum of complete bipartite graphs are graceful. They also proved that star of complete bipartite graph is graceful.

Singh [14] and Devaraj [15] defined a concept of triangular graceful labeling. Suppose that G is a graph with p vertices and q edges. We say that G is *triangular graceful* if there is an injection f from $V(G)$ to $\{0,1,2,...,T_q\}$ where T_q is the $q-th$ triangular number and the labels induced on each edge uv by $|f(u)-f(v)|$ are the first q triangular numbers. They proved that paths, caterpillars, unicyclic graphs are triangular graceful. It is also proved that wheels, helms and K_n with $n\geq 3$ are not triangular graceful. They also proved that all trees are triangular graceful.

The concept of super triangular graceful labeling was introduced by Ramachandran and Megala [16]. Let $G=(V,E)$ be a graph with p vertices and q edges. An injective function $f:V(G)\rightarrow\{0,1,2,...,S_q\}$ where S_q is the $q-th$ super triangular number is said to be a *super triangular graceful labeling* if the induced edge labeling $f^*(u,v)=|f(u)-f(v)|$ is a bijection onto the set $\{S_1,S_2,...,S_q\}$. A graph with a super triangular graceful labeling is called a super triangular graceful graph. They proved that any paths and Olive trees admits super triangular graceful labeling.

In the next section, we prove that $B_{n,n}$, $\langle B_{n,n}:w\rangle$ and $\langle B_{n,n}:P_m\rangle$ admits super triangular graceful labeling.

3. Super Triangular Graceful Labeling

Definition 3.1 K_2 with n pendant edges attached at each point is called a bistar and is denoted by $B_{n,n}$.

Theorem 3.1 The graph $G = B_{n,n}$ is a super triangular graceful graph $\forall n \geq 2$.

Proof. Let u and v be the vertices of K_2. Let $u_1, u_2, ..., u_n$ be the pendant vertices joined with the vertex u. Let $v_1, v_2, ..., v_n$ be the pendant vertices joined with the vertex v. Then $V(G) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{uv, uu_i, vv_i : 1 \leq i \leq n\}$. Thus, $|V(G)| = 2n+2$ and $|E(G)| = 2n+1$. Define $f : V(G) \to \{0, 1, 2, ..., S_{2n+1}\}$ by

$$f(u) = 0,$$

$$f(v) = S_{2n+1},$$

$$f(u_i) = S_{n+i}, 1 \leq i \leq n \text{ and}$$

$$f(v_i) = S_{2n+1} - S_i, 1 \leq i \leq n.$$

Now, $\max f(w) = \max\{0, S_{2n+1}, S_{n+i}, S_{2n+1} - S_i\}, 1 \leq i \leq n$

$= S_{2n+1}$, the $2n+1-th$ super triangular number.

Thus, it is clear that $f(w) \leq S_{2n+1} \forall w \in V(G)$. That is, $f(w) \in \{0, 1, 2, ..., S_{2n+1}\} \forall w \in V(G)$.

Next, we claim that f is injective. Since $\{f(u_i)\}$ is a strictly increasing and $\{f(v_i)\}$ is a strictly decreasing sequence. So, We have to show that $f(u_i) \neq f(v_i) \forall i$. For $i = 1, 2, ..., n$,

$$f(v_i) = S_{2n+1} - S_i$$
$$= \left(1^2 + 2^2 + 3^2 + ... + n^2 + ... + (2n)^2 + (2n+1)^2\right) - \left(1^2 + 2^2 + 3^2 + ... + i^2\right)$$
$$= (i+1)^2 + (i+2)^2 + (i+3)^2 + ... + n^2 + ... + (2n)^2 + (2n+1)^2$$

Therefore, $f(v_i)$ is not a super triangular number. But $f(u_i) = S_{n+i}$ is a super triangular number. That is, $f(u_i) \neq f(v_i) \forall i$. Hence all the vertex labels are distinct. Finally, we have to show that the edge labels are of the form $\{S_1, S_2, ..., S_{2n+1}\}$. Define the induced edge function $f^* : E(G) \to \{1, 2, ..., S_{2n+1}\}$ such that

$f^*(uv) = S_{2n+1}$,

$f^*(uu_i) = S_{n+i}, 1 \le i \le n$ and

$f^*(vv_i) = S_i, 1 \le i \le n$.

It is easily seen that f^* is injective and $f^*(E(G)) = \{S_1, S_2, ..., S_{2n+1}\}$. Hence the graph $B_{n,n}$ is a super triangular graceful graph $\forall n \ge 2$.

For example, the super triangular graceful labeling of $B_{10,10}$ is shown in Fig. 3.1.

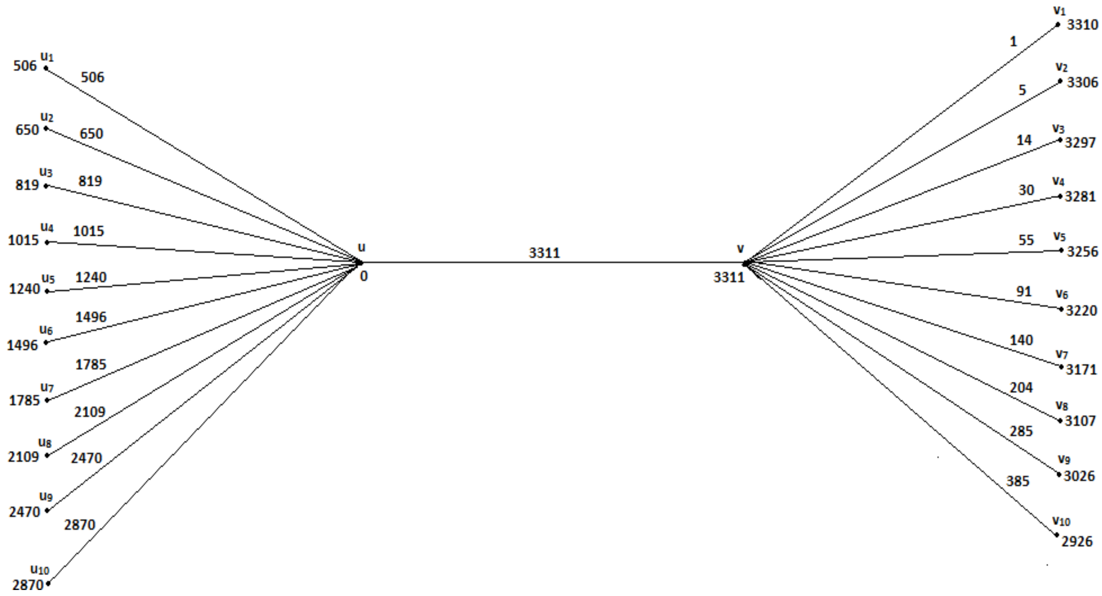

Fig. 3.1: The super triangular graceful labeling of $B_{10,10}$.

Definition 3.2 The graph $\langle B_{n,n} : w \rangle$ is the graph obtained by subdividing the central edge of the bistar $B_{n,n}$ with the vertex w.

Theorem 3.2 The graph $G = \langle B_{n,n} : w \rangle$ is a super triangular graceful graph $\forall n \ge 2$.

Proof. Let w be a vertex of subdivision of the central edge of $B_{n,n}$. Let u and v be the pendant vertices joined with the vertex w. Let $u_1, u_2, ..., u_n$ be the pendant vertices joined with the vertex u. Let $v_1, v_2, ..., v_n$ be the pendant vertices joined with the vertex v. Then $V(G) = \{u, v, w, u_i, v_i : 1 \le i \le n\}$ and $E(G) = \{uw, vw, uu_i, vv_i : 1 \le i \le n\}$. Thus, $|V(G)| = 2n+3$ and $|E(G)| = 2n+2$. Define $f : V(G) \to \{0, 1, 2, ..., S_{2n+2}\}$ by

$f(u)=0$,

$f(w)=S_{2n+2}$,

$f(v)=(2n+2)^2$

$f(u_i)=S_{n+i}, 1\le i\le n$ and

$f(v_i)=(2n+2)^2+S_i, 1\le i\le n$.

Now, $\max f(x)=\max\left\{0, S_{2n+2}, (2n+2)^2, S_{n+i}, (2n+2)^2+S_i\right\}, 1\le i\le n$

$= S_{2n+2}$

This implies that $\max f(x)$ is a $2n+2-th$ super triangular number. Thus, it is clear that $f(x)\le S_{2n+2} \forall x\in V(G)$. That is, f *. Next, we claim that f is injective. It is clear that $\{f(u_i)\}$ and $\{f(v_i)\}$ is a strictly increasing sequence. First, we have to show that $f(u_i)\ne f(v)\ne f(v_i)\forall i$. Obviously, $f(v)< f(v_i)\forall i$ .Also,

$$\begin{aligned} f(v) &= (2n+2)^2 \\ &= 4(n+1)^2 \\ &= 4(n+1)(n+1) \\ &= 2(2n+2)(n+1) \\ &= 2(2n+1)\left(\frac{1}{2n+1}\right)(n+1) \\ &= \frac{n(n+1)(2n+1)}{6}\left[\frac{12}{n}\left(1+\frac{1}{2n+1}\right)\right] \end{aligned}$$

Now, we prove that $\frac{12}{n}\left(1+\frac{1}{2n+1}\right)\ne 1 \quad \forall n$. Suppose that $\frac{12}{n}\left(1+\frac{1}{2n+1}\right)=1$ for some n.

Then $2n^2-23n-24=0$.

$\Rightarrow n\notin \mathbb{N}$

$\Rightarrow \frac{12}{n}\left(1+\frac{1}{2n+1}\right)\ne 1 \quad \forall n$

$\Rightarrow f(v)$ is not a super triangular number.

$\Rightarrow f(v)\ne f(u_i)\forall i$

$\Rightarrow f(u_i) \neq f(v) \neq f(v_i) \forall i$.

Next, we have to show that $f(u_i) \neq f(v_i) \forall i$.

For $i = 1,2,...,n$,

$$f(v_i) = S_i + (2n+2)^2$$
$$= 1^2 + 2^2 + 3^2 + ... + i^2 + (2n+2)^2.$$

Suppose that $(2n+2)^2 = (i+1)^2, i \leq n$. Then $2n+2 = i+1$

$\Rightarrow 2(n+1) = i+1$

$\Rightarrow 2(i+j+1) = i+1$ where $n = i+j, j = 0,1,2,...,n-1$

$\Rightarrow 2(i+1)\left(1+\frac{j}{i+1}\right) = i+1$, which is not possible because $2\left(1+\frac{j}{i+1}\right) \neq 1$.

$\Rightarrow f(v_i)$ is not a super triangular number.

But each $f(u_i)$ is a super triangular number. Thus, $f(u_i) \neq f(v_i) \forall i$. Hence all the vertex labels are distinct.

Finally, we have to show that the edge labels are of the form $\{S_1, S_2, ..., S_{2n+2}\}$. Define the induced edge function $f^*: E(G) \to \{1, 2, ..., S_{2n+2}\}$ such that

$$f^*(uw) = S_{2n+2},$$

$$f^*(vw) = S_{2n+1},$$

$$f^*(uu_i) = S_{n+i}, 1 \leq i \leq n \text{ and}$$

$$f^*(vv_i) = S_i, 1 \leq i \leq n.$$

It is easily seen that f^* is injective and $f^*(E(G)) = \{S_1, S_2, ..., S_{2n+2}\}$. Hence the graph $\langle B_{n,n} : w \rangle$ is a super triangular graceful graph $\forall n \geq 2$.

For example, the super triangular graceful labeling of $\langle B_{10,10} : w \rangle$ is shown in Fig. 3.2.

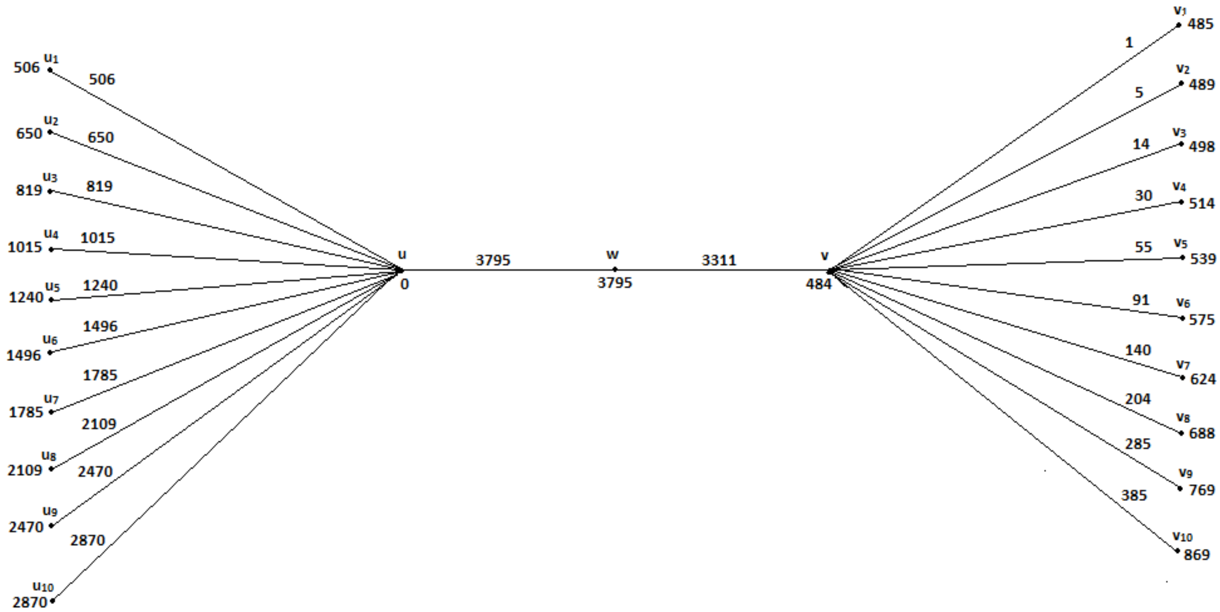

Fig. 3.2: Super triangular graceful labeling of $\langle B_{10,10} : w \rangle$.

Definition 3.3 The graph $\langle B_{n,n} : P_m \rangle$ is the graph obtained by subdividing the central edge of the bistar $B_{n,n}$ with path P_m.

Theorem 3.3 The graph $G = \langle B_{n,n} : P_m \rangle$ is a super triangular graceful graph $\forall n \geq 2$.

Proof. Let $\{w_1, w_2, ..., w_m\}$ be a vertex set of P_m. Let u be the pendant vertex joined with the vertex w_1. Let v be the pendant vertex joined with the vertex w_m. Let $u_1, u_2, ..., u_n$ be the pendant vertices joined with the vertex u. Let $v_1, v_2, ..., v_n$ be the pendant vertices joined with the vertex v. Then $V(G) = \{u, v, u_i, v_i, w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(G) = \{uw_1, vw_m, uu_i, vv_i, w_j w_{j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$. Thus, $|V(G)| = 2n + m + 2$ and $|E(G)| = 2n + m + 1$. Define $f : V(G) \rightarrow \{0, 1, 2, ..., S_{2n+m+1}\}$ by

$$f(u) = 0,$$

$$f(w_1) = S_{n+m+1},$$

$$f(w_i) = \begin{cases} f(w_{i-1}) - S_{n+m-i+2} & : i \text{ is even}, 2 \leq i \leq m \\ f(w_{i-1}) + S_{n+m-i+2} & : i \text{ is odd}, 2 \leq i \leq m \end{cases}$$

$$f(v) = \begin{cases} f(w_m) + S_{n+1} & : m \text{ is even} \\ f(w_m) - S_{n+1} & : m \text{ is odd} \end{cases}$$

$f(u_i) = S_{n+m+i+1}, 1 \le i \le n$ and

$f(v_i) = f(v) - S_i, 1 \le i \le n$.

It is easy to check that the labels of the edges of the graph are $S_1, S_2, ..., S_{2n+m+1}$. That is, we define the induced edges function $f^*: E(G) \to \{1, 2, ..., S_{2n+m+1}\}$ such that

$$f^*(uw_1) = S_{m+n+1},$$

$$f^*(vw_m) = S_{n+1},$$

$$f^*(w_i w_{i+1}) = S_{n+m-i+1},\ 1 \le i \le m-1,$$

$$f^*(uu_i) = S_{n+m+i+1},\ 1 \le i \le n \text{ and}$$

$$f^*(vv_i) = S_i,\ 1 \le i \le n .$$

Hence the graph $\langle B_{n,n} : P_m \rangle$ is a super triangular graceful graph $\forall n \ge 2$.

For example, the super triangular graceful labeling of $\langle B_{10,10} : P_7 \rangle$ is shown in Fig. 3.3.

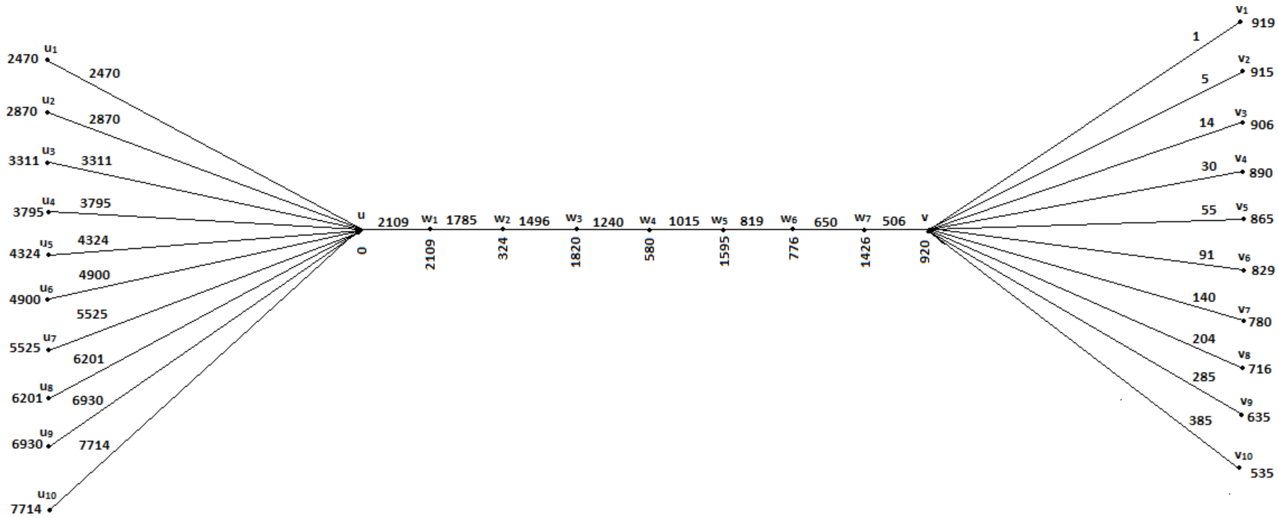

Fig. 3.3: The super triangular graceful labeling of $\langle B_{10,10} : P_7 \rangle$.

4. Conclusion

Nowadays graph labeling has much attention from different brilliant researches in graph theory which has rigorous applications in many disciplines such as communication networks, coding theory, optimal circuits layouts, astronomy, radar and graph decomposition problems. In this paper specially, we studied the two types of labeling namely, prime labeling and combination labeling and their properties.

An extensive review of certain studies on the super triangular graceful labeling is also done. It is shown that super triangular graceful labeling for $B_{n,}$, $(B_{n,n}: w)$ and $(B_{n,n}: P_m)$. In our future work we plan to extend this idea to some more graph operations.

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